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Distributional Boundary Values of Harmonic and Analytic Functions

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A theory for distributional boundary values of harmonic and analytic functions is presented. In this analysis there arise several indicators that measure the growth of these functions near the boundaries. An extension of the Phragmén–Lindelöf maximum principle is derived. Furthermore, the algebraic properties of the space of real periodic distributions are studied. By introducing a new product, the harmonic product, the boundary conditions involving harmonic functions are transformed into ordinary differential equations.

1. INTRODUCTION

There are several areas of common interest in the theory of distributions and the theories of harmonic and analytic functions. So far, attention has been focused on the generalized analytic functions and on the Cauchy representation of the distributions [1–3]. In the classical theory, the behavior of the harmonic and analytic functions at the boundaries was first studied by Fatou [4] and has since been a topic of an extensive research. The reference to this literature is available in [5, 6]. In this study we present the theory of distributional boundary values of harmonic functions defined in the unit ball of $\mathbb{R}^n$ and those of analytic functions defined in the unit disk.

In Section 2 we introduce several indicators that measure the growth of a harmonic function near the boundary. We find several relations among different indicators which enable us to characterize the harmonic functions that have distributional boundary values. This characterization is subsequently used to derive the uniqueness and boundedness properties. Section 3 is devoted to a consideration of the basic properties of analytic functions that have distributional boundary values. We prove an extension of the Phragamén–Lndelöf maximum principle. We study the order of the boundary distribution and show that our class of functions is much larger than the

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Hardy spaces H^p. We also prove that the logarithm of a zero-free function of this class also belongs to it.

In Section 4 we derive jump relations for the sectionally holomorphic functions. We then study the algebraic properties of $\mathscr{D}'(S_1)$, the space of real periodic distributions in one variable. A new product, the harmonic product, is introduced and the relationship with the ordinary product is obtained. This product is then used to transform the boundary conditions involving harmonic functions into ordinary differential equations. We present some examples. In particular, we find the explicit solution for non-normal boundary conditions. These techniques are extended to the complex distributions in the last section.

2. Boundary Values in S_{n-1}

Let B_n be the open unit ball in $\mathbb{R}^n$ and S_{n-1} its boundary, the unit sphere (we shall denote them by B and S if n is clear from the context). Any point x in B can be represented as $x = ry$ with $0 \leqslant r < 1$, $y \in S$. We shall denote by $P(x, y)$ the Poisson kernel, given by

$$P(x, y) = \frac{1 - |x|^2}{\omega_n |x - y|^n}, \qquad x \in B, \quad y \in S, \tag{2.1}$$

where ω_n is the area of S. Let H_k denote the space of spherical harmonics of degree k, that is, the set of restrictions of harmonic homogeneous polynomials of degree k to S. The space of Lebesgue square integrable functions can be decomposed as

$$L^2(S) = \sum_{k=0}^{\infty} H_k, \tag{2.2}$$

where the sum on the right is an orthogonal direct sum. The dimension of H_k, denoted by μ_k, is given by [7, 8]

$$\mu_k = \frac{(2k + n - 2)(k + n - 3)!}{k!\,(n - 2)!}. \tag{2.3}$$

The space H_k admits a reproducing kernel $Z_k(x, y)$, characterized by the property

$$f(x) = \int_S f(y)\, Z_k(x, y)\, d\sigma(y), \qquad x \in S, \quad f \in H_k, \tag{2.4}$$

where $d\sigma = (1/\omega_n)\, dS$ is the normalized measure on the sphere. This reproducing kernel is real valued, symmetric, and invariant under rotations of the sphere. The zonal harmonic of degree k with pole at x is the function Z_k^x given by $Z_k^x(y) = Z_k(x, y)$. Observe that

$$f(x) = \langle f, Z_k^x \rangle, \qquad f \in H_k. \tag{2.5}$$

It can be shown that

$$\|Z_k^x\|_2^2 = \|Z_k^x\|_\infty = Z_k(x, x) = \mu_k. \tag{2.6}$$

We shall need the following expansion [8] in the sequel:

$$P(rx, y) = \sum_{k=0}^{\infty} r^k Z_k(x, y), \qquad x, y \in S, \quad 0 \leqslant r < 1. \tag{2.7}$$

Lemma 1. *Let $f \in H_k$, then*

$$\|f\|_p \leqslant \|f\|_q, \qquad 1 \leqslant p \leqslant q \leqslant \infty, \tag{2.8}$$

$$\|f\|_q \leqslant \|f\|_p \mu_k^{((1/p)-(1/q))}, \qquad 1 \leqslant p \leqslant 2, \quad p \leqslant q \leqslant \infty. \tag{2.9}$$

Proof. The first inequality follows from Hölder's inequality:

$$\|f\|_p = \left[\int |f|^p \, d\sigma\right]^{1/p} \leqslant \left\{\left[\int |f|^q \, d\sigma\right]^{p/q} \left[\int_1^r 1^r \, d\sigma\right]^{1/r}\right\}^{1/p} = \|f\|_q, \tag{2.10}$$

where $1/r + p/q = 1$. For the second we first consider the case $q = \infty$. We have

$$|f(x)| = |\langle Z_k^x, f \rangle| \leqslant \|Z_k^x\|_r \|f\|_p, \tag{2.11}$$

where $1/p + 1/r = 1$. If $1 \leqslant p \leqslant 2$, then $2 \leqslant r \leqslant \infty$, the Hölder's inequality gives

$$\|Z_k^x\|_r^r \leqslant \|Z_k^x\|_\infty^{r-2} \|Z_k^x\|_2^2 \leqslant \mu_k^{r-2} \mu_k = \mu_k^{r-1}, \tag{2.12}$$

and since $(r-1)/r = 1/p$, (2.11) gives

$$\|f\|_\infty \leqslant \mu_k^{1/p} \|f\|_p. \tag{2.13}$$

Finally, for arbitrary q, $q \geqslant p$, we have

$$\begin{aligned}
\|f\|_q = \left[\int |f|^q \, d\sigma\right]^{1/q} &= \left[\int |f|^{q-p} |f|^p \, d\sigma\right]^{1/q} \\
&\leqslant (\|f\|_\infty^{q-p})^{1/q} (\|f\|_p)^{p/q} \\
&\leqslant \mu_k^{(1/p)(q-p)(1/q)} \cdot \|f\|_p^{1-(p/q)} \cdot \|f\|_p^{p/q} \\
&\leqslant \mu_k^{((1/p)-(1/q))} \|f\|_p. \qquad \blacksquare
\end{aligned}$$

According to (2.2), each function $f \in L^2(S)$ can be expanded as

$$f = \sum_{k=0}^{\infty} f_k, \tag{2.14}$$

where $f_k \in H_k$ are its projections onto H_k, given by

$$f_k(x) = \int_S f(y)\, Z_k(x, y)\, d\sigma(y) = \langle f, Z_k^x \rangle. \tag{2.15}$$

Let $\mathscr{D}(S)$ be the space of real valued infinitely differentiable functions in S and $\mathscr{D}'(S)$ its dual, the space of distributions on S. The evaluation of the distribution t at the test function ϕ is denoted by $\langle t, \phi \rangle$ since it reduces to their inner product if $t \in L^2(S)$. Given $t \in \mathscr{D}'$, its projection onto H_k is the function $t_k(x) = \langle t, Z_k^x \rangle$.

LEMMA 2. *Let $\phi \in L^2(S)$ and ϕ_k its projection onto H_k, then $\phi \in \mathscr{D}(S)$ if and only if for $1 \leqslant p \leqslant \infty$ and $\alpha > 0$,*

$$\lim_{k \to \infty} \|\phi_k\|_p \, k^{\alpha} = 0. \tag{2.16}$$

Let $t \in \mathscr{D}'(S)$ and t_k its projection onto H_k, then $t_k \in H_k$ and

$$t = \sum_{k=0}^{\infty} t_k, \tag{2.17}$$

in the distributional sense. Furthermore, there exists a real number β such that

$$\lim_{k \to \infty} \|t_k\|_p \, k^{-\beta} = 0, \tag{2.18}$$

for all p, $1 \leqslant p \leqslant \infty$. Conversely, if $\{t_k\}$ is a sequence of elements of H_k that satisfy (2.18) *for some β and some p, then the series* (2.17) *converges in $\mathscr{D}'$.*

Proof. Let $\phi \in L^2(S)$ be any function satisfying (2.16). We shall show that if L is any differential operator of degree one then $L\phi$ exists and satisfies (2.16). This will imply that $\phi \in \mathscr{D}$. It suffices to establish the result for $L = \partial/\partial x_i$.

Let $p(x)$ be a polynomial of degree at most k in one variable. Let $x_1, \ldots, x_r$ be the zeros of $p(x)$ that belong to $[a, b]$ and set $x_0 = a$, $x_{r+1} = b$. Then

$$\begin{aligned} \int_a^b |p'(x)|\, dx &= \sum_{j=0}^{r} \int_{x_j}^{x_{j+1}} |p'(x)|\, dx \\ &= \sum_{j=0}^{r} |p(x_{j+1}) - p(x_j)| \\ &\leqslant 2(r+1) \sup\{|p(x)| : x \in [a, b]\}, \end{aligned}$$

so that

$$\int_a^b |p'(x)|\,dx \leqslant 2k \sup\{|p(x)|: x \in [a, b]\}. \tag{2.19}$$

Let $f \in H_k$, then if $x_2, \ldots, x_n$ are fixed, f is a polynomial of degree at most k in x_1; we can therefore apply (2.19) to obtain

$$\int_S \left|\frac{\partial f}{\partial x_i}\right| d\sigma \leqslant Ck\,\|f\|_\infty, \qquad f \in H_k, \tag{2.20}$$

for certain constant C that depends only on n. Now we appeal to the results of Lemma 1 and deduce the convergence of the series

$$\sum_{k=0}^{\infty} \partial\phi_k/\partial x_1. \tag{2.21}$$

Thus, $\partial\phi/\partial x_1$ exists and its projections onto H_k also satisfy (2.16).

Conversely, let $\phi \in \mathscr{D}$. Let L be the differential operator $L = \sum_{i=1}^n x_i\,\partial/\partial x_i$. Then $Lf = kf$ if $f \in H_k$. Since $L^j\phi \in L^2(S)$ for any $j = 1, 2, 3, \ldots$, relation (2.16) follows at once.

Let now $t \in \mathscr{D}'$. Then t restricted to H_k is a continuous linear function on H_k, so that there is an element $g_k \in H_k$ such that

$$\langle t, f\rangle = \langle f, g_k\rangle, \qquad f \in H_k. \tag{2.22}$$

By taking $f = Z_k^x$ we obtain

$$t_k(x) = \langle t, Z_k^x\rangle = \langle Z_k^x, g_k\rangle = g_k(x), \tag{2.23}$$

so that $g_k(x)$ is just the projection of t onto H_k as defined above.

The equation $t = \sum_{k=0}^{\infty} t_k$ is clearly valid in every H_k, so it is, a fortiori, in all $\mathscr{D}$.

If for all real numbers β,

$$\overline{\lim}\, \|t_k\|_2\, k^{-\beta} > 0, \tag{2.24}$$

we could obtain a sequence of indices $k_1 < k_2 < k_3 < \cdots$ such that

$$\|t_k\|_2 > k_j^j, \qquad j = 1, 2, 3, \ldots. \tag{2.25}$$

Also, by letting

$$\phi = \sum_{j=1}^{\infty} \frac{1}{\|t_{k_j}\|_2\, k_j^j}\, t_{k_j}, \tag{2.26}$$

we would obtain $\phi \in \mathscr{D}$. But

$$\langle t, \phi \rangle = \sum_{j=1}^{\infty} \frac{\|t_{k_j}\|_2}{k_j^j} = \infty, \tag{2.27}$$

which is a contradiction. Thus, $\lim_{k \to \infty} \|t_k\| k^{-\beta} = 0$ for some β, and, according to Lemma 1, (2.18) will also hold for any p.

Conversely, if $\{t_k\}$ is a sequence of elements of H_k satisfying (2.18) it follows that the series

$$\sum_{k=0}^{\infty} \langle t_k, \phi \rangle,$$

converges for every $\phi \in \mathscr{D}$. Hence, $\sum_{k=0}^{\infty} t_k$ is an element t of $\mathscr{D}'$ and it is easy to verify that the projection of t onto H_k is precisely t_k. ∎

Now let $\{a_k\}$ be a sequence of non-negative numbers. The extended real number

$$d[\{a_k\}] = \overline{\lim_{k \to \infty}} \frac{\log a_k}{\log k} \tag{2.28}$$

is the order of growth of the sequence

$$\begin{aligned} d[\{a_k\}] = \overline{\lim_{k \to \infty}}\, a_k k^{-\beta} &= \infty, \qquad \text{if} \quad \beta < d[\{a_k\}], \\ &= 0, \qquad \text{if} \quad \beta > d[\{a_k\}]. \end{aligned} \tag{2.29}$$

Let $\{f_k\}$ be a sequence of elements of H_k and $f = \sum f_k$. Then $f \in \mathscr{D}(S)$ if and only if $d[\{\|f_k\|_p\}] = -\infty$, while $f \in \mathscr{D}'(S)$ if and only if $d[\{\|f_k\|_p\}] < \infty$.

Any function $U(x)$, harmonic in B, can be expanded as

$$U(ry) = \sum_{k=0}^{\infty} r^k u_k(y), \tag{2.30}$$

where $u_k \in H_k$. We shall denote

$$d_p(U) = d[(\|u_k\|_p)], \tag{2.31}$$

for $1 \leqslant p \leqslant \infty$. Since

$$\lim_{k \to \infty} \mu_k k^{2-n} = 2/(n-2)!, \tag{2.32}$$

we obtain the following theorem with the aid of Lemma 1.

THEOREM 1. *Let U be harmonic in B_n. Then*

$$d_p(U) \leqslant d_q(U), \qquad 1 \leqslant p \leqslant q \leqslant \infty, \tag{2.33}$$

$$d_q(U) \leqslant d_p(U) + (n-2)(1/p - 1/q), \qquad 1 \leqslant p \leqslant 2, \quad p \leqslant q \leqslant \infty. \ ∎ \tag{2.34}$$

Next, by combining this result with Lemma 2 we obtain

THEOREM 2. *Given a harmonic function U on B, the limit*

$$\lim_{r\to 1} U(ry) = u(y), \qquad y \in S \tag{2.35}$$

exists in the distributional sense if and only if $d_p(U)$ is finite for some p. For each $u \in \mathscr{D}'(S)$ there exists a unique harmonic function U on B that satisfies (2.35); U is given by

$$U(x) = \langle u(y), P(x, y)\rangle, \qquad x \in B. \tag{2.36}$$

Proof. Let us expand U as in (2.30) above and let

$$u = \sum_{k=0}^{\infty} u_k. \tag{2.37}$$

If $d_p(U)$ is finite, then $u \in \mathscr{D}'(S)$. Let $\phi \in \mathscr{D}(S)$ with projections ϕ_k onto H_k. Then

$$\langle U(ry), \phi(y)\rangle = \sum_{k=0}^{\infty} r^k \langle u_k, \phi_k\rangle, \tag{2.38}$$

is a series that converges at $r = 1$ and hence by Abel's limit theorem,

$$\lim_{r\to 1} \langle U(ry), \phi(y)\rangle = \sum_{k=0}^{\infty} \langle u_k, \phi_k\rangle = \langle u, \phi\rangle. \tag{2.39}$$

Since ϕ was arbitrary, (2.35) follows at once.

Let now $u \in \mathscr{D}'(S)$. To prove existence we expand u as in (2.37) and define U as in (2.30). Clearly, the distributional limit of this harmonic function, as $r \to 1$, is u.

To prove uniqueness, let U be a harmonic function such that

$$\lim_{r\to 1} U(ry) = 0, \qquad \text{in} \quad \mathscr{D}', \tag{2.40}$$

then

$$\|u_k\|_2^2 = \lim_{r\to 1} \langle U(ry), u_k\rangle = 0 \tag{2.41}$$

for all k, and hence $U \equiv 0$.

Finally, with the help of the expansion (2.7) we obtain

$$\langle u(y), P(rx, y)\rangle = \sum_{k=0}^{\infty} r^k \langle u(y), Z_k(x, y)\rangle = U(y),$$

and (2.36) follows readily. ∎

DEFINITION. A harmonic function U in B such that $\lim_{r\to 1} U(ry)$ exists in $\mathscr{D}'(S)$ is called a regular harmonic function.

Let U be regular harmonic, then as r approaches 1, U will in general not remain bounded. However, even though U may become very large, its growth is relatively slow, where the slowness is measured by the following indicators. If we use the notation

$$\begin{aligned} M_p(r, U) &= \left[\int_S |U(ry)|^p \, d\sigma(y)\right]^{1/p}, \qquad 1 \leqslant p < \infty, \\ &= \max\{|U(ry)|: y \in S\}, \qquad p = \infty, \end{aligned} \tag{2.42}$$

then the indicator is

$$C_p(U) = -\varliminf_{r\to 1} \frac{\log M_p(r, U)}{\log(1-r)}. \tag{2.43}$$

We shall see that the regular harmonic functions are precisely those of slow growth, i.e., those for which $C_\infty(U)$, or any $C_p(U)$, is finite. This follows from our next result:

THEOREM 3. *Let U be any harmonic function in B. Then*

$$d_\infty(U) \leqslant C_1(U) + n - 2 \leqslant C_\infty(U) + n - 2 \leqslant d_\infty(U) + n - 1. \tag{2.44}$$

Proof. We have

$$u_k(x) = (1/r^k)\langle U(ry), Z_k^x(y)\rangle. \tag{2.45}$$

If $C_p(U) < \beta$, then for some constant N we have $M_p(r, U) \leqslant N(1-r)^{-\beta}$, and so

$$|u_k(x)| \leqslant (N/r^k(1-r)^\beta) \, \|Z_k^x\|_q, \tag{2.46}$$

where $1/p + 1/q = 1$, for all r, $0 \leqslant r < 1$. By taking $r = k/(k+\beta)$, we obtain

$$|u_k(x)| \leqslant Nk^\beta[1 + \beta/k]^{k+\beta} \beta^{-\beta} \, \|Z_k^x\|_q. \tag{2.47}$$

If $1 \leqslant p \leqslant 2$, then $\|Z_k^x\|_q \leqslant \mu_k^{1/p}$ and hence

$$d_\infty(U) \leqslant \beta + ((n-2)/p), \tag{2.48}$$

and since β is arbitrary,

$$d_\infty(U) \leqslant C_p(U) + ((n-2)/p), \qquad 1 \leqslant p \leqslant 2. \tag{2.49}$$

For the second inequality, let us suppose that $\beta > \max\{d(U), -1\}$. There exists a constant N_1 such that

$$\|u_k\|_\infty \leqslant \frac{N_1 \Gamma(\beta + k + 1)}{\Gamma(\beta + 1) \Gamma(k + 1)}. \tag{2.50}$$

Hence

$$|U(ry)| \leqslant \sum_{k=0}^{\infty} \|u_k\|_\infty r^k \leqslant \frac{N_1}{(1 - r)^{\beta + 1}}, \tag{2.51}$$

so $\beta + 1 \geqslant C_\infty(U)$, and we have

$$C_\infty(U) \leqslant d_\infty(U) + 1, \qquad \text{if} \quad d_\infty(U) \geqslant -1, \tag{2.52}$$

$$C_\infty(U) = 0, \qquad \text{if} \quad d_\infty(U) \leqslant -1. \quad \blacksquare \tag{2.53}$$

Remark. Other relations among the C_p's and d_p's are easy to derive. For example, $C_q \leqslant C_p + (n - 1)(1/p - 1/q)$, $1 \leqslant p \leqslant q \leqslant \infty$. When $n = 2$, the Hausforf–Young inequality [9] can be used to prove that $C_q \leqslant d + 1 - 1/q$, for $2 \leqslant q \leqslant \infty$ (observe that $d = d_1 = \cdots = d_\infty$ in this case). In fact, let p be such that $1/p + 1/q = 1$, then

$$M_q(r, U) = \left[\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{-\infty}^{\infty} a_k e^{ik\theta} r^{|k|} \right|^q \right]^{1/q} \leqslant \left[\sum_{-\infty}^{\infty} |a_k r^{|k|}|^p \right]^{1/p}$$

$$\leqslant N \left[\sum_0^\infty \frac{\Gamma(p\beta + k + 1)\, r^{k\beta}}{\Gamma(p\beta + 1) \Gamma(k + 1)} \right]^{1/p} = N(1 - r^p)^{-(\beta + 1/p)},$$

for $\beta > \max\{d, 1/q - 1\}$ and some constant N. This gives the result.

We immediately obtain

THEOREM 4. *Necessary and sufficient condition for the harmonic function U to be the regular harmonic is that $C_p(U)$ be finite for some p, $1 \leqslant p \leqslant \infty$.* $\blacksquare$

A classical result states that any positive harmonic function in the unit ball has as its limit a positive measure. We can generalized this result as follows. Let U be harmonic in B and let

$$U_+(x) = \max\{U(x), 0\}, \tag{2.54}$$

$$U_-(x) = \min\{U(x), 0\}, \tag{2.55}$$

then we have the following theorem.

THEOREM 5. *Let U be harmonic in B. Suppose there are constants N and β such that for $0 \leqslant r < 1$, $y \in S$,*

$$-N/(1-r)^{\beta} \leqslant U(ry). \tag{2.56}$$

Then U is regular harmonic.

Proof. We may suppose that $U(0)=0$. By the mean value property,

$$0 = \int_S U(ry)\, d\sigma(y) = \int_S U_+ \, d\sigma + \int_S U_- \, d\sigma. \tag{2.57}$$

Hence

$$M_1(r, U) = \int_S |U| \, d\sigma = \int_S (U_+ - U_-) \, d\sigma = -2 \int_S U_- \, d\sigma \leqslant 2N/(1-r)^{\beta}, \tag{2.58}$$

and therefore $C_1(U) \leqslant \beta$. ∎

The space of regular harmonic functions is isomorphic to the space of distributions on S and under this isomorphism it becomes a nuclear topological vector space. A more intrinsic topology is constructed as follows. Let $N_{\alpha,p}$ be defined as

$$N_{\alpha,p}(U) = \left[\int_0^1 (1-r)^{\alpha} M_p^p(r, U) \, dr \right]^{1/p}, \tag{2.59}$$

and let $A_{\alpha,p}$ be the set of harmonic functions for which $N_{\alpha,p}$ is finite. Then $N_{\alpha,p}$ is a norm in $A_{\alpha,p}$ and the space of regular harmonic functions is given as

$$A = \bigcup_{\alpha > 0} A_{\alpha,p}, \tag{2.60}$$

and can be given the inductive limit topology. As is clear from the foregoing discussion, this topology is equivalent to the one induced by $\mathscr{D}'(S)$.

The explicit correspondence between A and $\mathscr{D}'(S)$ enables us to derive some basic results.

It is convenient to have a standard representation for differential operators on S. For this purpose we shall write first order operators in $\mathscr{D}$ or $\mathscr{D}'$ as linear combinations of the basic operators,

$$\frac{\delta}{\delta x_i} = \sum_{j=1}^{n} (\delta_{ij} - x_i x_j) \frac{\partial}{\partial x_j}, \qquad i = 1,..., n. \tag{2.61}$$

Let $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $|\alpha|=\alpha_1+\cdots+\alpha_n$, then D^α is the differential operator of order $|\alpha|$ and is defined as

$$D^\alpha=\frac{\delta^{|\alpha|}}{\delta x_1^{\alpha_1}\cdots\delta x_n^{\alpha_n}}. \tag{2.62}$$

The higher order operators will be written as linear combinations of these operators.

Let $\delta(y-\xi)$ be the Dirac delta function with source point at ξ:

$$\langle\delta(y-\xi),\phi(y)\rangle=\phi(\xi).$$

The form of these distributions on a point, or more generally, on a discrete set E is well known [2], namely,

$$t=\sum_{\xi\in E}\sum_{|\alpha|\leqslant m}a_\alpha(\xi)\,D^\alpha\delta(y-\xi),$$

where $a_\alpha(\xi)$ are constants. This result can be generalized as follows:

LEMMA 3. *Let t be a distribution whose support is a closed denumerable set E. Then there exists an integer k and constants $a_\alpha(\xi)$ for $\xi\in E$ and for any multiindex $\alpha=(\alpha_1\cdots\alpha_n)$ with $|\alpha|=\alpha_1+\cdots+\alpha_n\leqslant k$ such that*

$$t=\sum_{\xi\in E}\sum_{|\alpha|\leqslant k}a(\xi)\,D^\alpha\delta(y-\xi). \tag{2.63}$$

Proof. If the set E is finite, the result is known. Accordingly, let us suppose that E is infinite. Let E_1 be the set of isolated points of E. An easy application of the Baire theorem shows that E_1 is also infinite. We can define, by induction, the set E_m as the set of isolated points of $E\backslash(U_{j<m}E_j)$. Each set E_m is either infinite or empty and, since E is denumerable, we shall have $E=\bigcup_{j<m}E_j$, for some denumerable ordinal m.

Let us consider t as a distribution on $U_1=(S\backslash E)\cup E_1$. Its support on U_1 will be E_1, which is discrete in U_1, so that

$$t=t_1=\sum_{\xi\in E_1}\sum_{|\alpha|\leqslant k}a_\alpha(\xi)\,D^\alpha(y-\xi), \qquad \text{on} \quad U_1. \tag{2.64}$$

Thus $t-t_1$ vanishes on E_1, so that its support is contained in $E\backslash E_1$. We can repeat this argument to get a second distribution t_2, i.e., the sum of the derivatives of the delta functions of the points of E_2, such that the support of $t-t_1-t_2$ is contained in $E\backslash(E_1\cup E_2)$ and so on. After a denumerable number of steps we shall arrive at the desired expansion. ∎

THEOREM 6. *Let U be a non-vanishing regular harmonic function. Suppose that there exists a closed denumerable set E of S_n such that*

$$\lim_{\substack{x \to y \\ x \in B}} U(x) = 0, \qquad y \in S \backslash E. \tag{2.65}$$

Then $k = C_\infty(U) + 1 - n$ is a non-negative integer and there exist constants $a_\alpha(\xi)$ for $\xi \in E$ and for any multiindex $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_n \leqslant k$, such that

$$U(ry) = \sum_{\xi \in E} \sum_{|\alpha| \leqslant k} a_\alpha(\xi) D_y^\alpha P(ry, \xi). \tag{2.66}$$

Proof. Let $u(y) = \lim_{r \to 1} U(ry)$ be the limit distribution. Condition (2.65) implies that u vanishes on $S \backslash E$. We can therefore write

$$u = \sum_{\xi \in E} \sum_{|\alpha| \leqslant m} a_\alpha(\xi) \bar{D}_y^\alpha \delta(y - \xi), \tag{2.67}$$

for some constants $a_\alpha(\xi)$ and some integer m where $\delta(y - \xi)$ is the Dirac delta function with source point ξ. Using (2.36) we obtain

$$U(ry) = \sum_{\xi \in E} \sum_{|\alpha| \leqslant m} a_\alpha(\xi) D_y^\alpha P(ry, \xi). \tag{2.68}$$

But $C_\infty(D_y^\alpha P) = |\alpha| + n - 1$ and thus by taking $m = k$ minimal we get $C_\infty(U) = k + n - 1$, as desired. ∎

THEOREM 7. *Let U be a regular harmonic function. Suppose that there is a closed denumerable set E such that*

$$\overline{\lim_{\substack{x \to y \\ x \in B}}} |U(x)| \leqslant M, \qquad y \in S \backslash E. \tag{2.69}$$

If $C_\infty(U) < n - 1$, then

$$|U(x)| \leqslant M, \qquad x \in B. \tag{2.70}$$

Proof. Let $u = \lim_{r \to 1} U(ry)$. Condition (2.69) implies that on $S \backslash E$, u is bounded measure, while at points of E, u will be a sum of delta functions. We can then write $U = U_1 + U_2$, where U_1 is bounded by M and where U_2 vanishes at $S_n \backslash E$. But according to the previous theorem, $C_\infty(U) = C_\infty(U_2) \geqslant n - 1$ unless $U_2 = 0$ and so $U = U_1$is bounded by M. ∎

A similar argument shows that

THEOREM 8. *Let U be regular harmonic. Suppose there is a constant $\alpha < n-1$ and a closed denumerable set E such that*

$$\lim_{\substack{x \to y \\ x \in B}} (1-|x|)^{\beta} U(x) = 0, \qquad y \in S \backslash E, \quad \beta > \alpha. \tag{2.71}$$

If $C_{\infty}(U) < n-1$, then $C_{\infty}(U) \leqslant \alpha$. ∎

From now on we consider the case $n=2$. We shall use the parametrization $z = e^{i\theta}$, $0 \leqslant \theta < 2\pi$, to describe S. Since all d_p's coincide in this case we shall use the simple notation $d(U)$.

We consider the pointwise convergence of $\lim_{r \to 1} U(re^{i\theta})$. A classical theorem of Fatou states that this limit exists almost everwhere if U is positive or, more generally, if its boundary value is a measure, i.e., U is the difference of two positive harmonic functions. As we shall see in the next section, however, the pointwise limit does not exist almost everywhere for almost all regular harmonic functions not of the above type. Other type of poblems are illustrated by the following example. Let

$$U(re^{i\theta}) = \frac{2r(r^2-1)\sin\theta}{2\pi(1-2r\cos\theta + r^2)^2}, \tag{2.72}$$

then pointwise,

$$\lim_{r \to 1} U(re^{i\theta}) = 0, \qquad 0 \leqslant \theta \leqslant 2\pi.$$

On the other hand, it is easy to see that the boundary value of U is $\delta'(\theta)$.

As the next theorem shows, some insight into the boundary value of a regular harmonic function can be gained from the pointwise limit.

THEOREM 9. *Let U be a regular harmonic function. Suppose*

$$\lim_{r \to 1} U(re^{i\theta}) = 0, \tag{2.73}$$

for all θ, $0 \leqslant \theta \leqslant 2\pi$, and that the convergence is uniform in the compacts of $S \backslash E$ where E is a closed denumerable set. Suppose that at each point ξ of E there are l different numbers $\theta_1, \dots, \theta_l \in \,]\pi/2, 3\pi/2[$ such that

$$\lim_{\varepsilon \to 0} U(\xi + \varepsilon e^{i\theta_n}) = 0, \qquad 1 \leqslant n \leqslant l. \tag{2.74}$$

If $C_{\infty}(U) < l+1$, then $U = 0$.

Proof. Using Theorem 6 we can write

$$U(re^{i\theta}) = \sum_{\xi \in E} \sum_{j=0}^{k} a_j(\xi) \frac{\partial^j P}{\partial \theta^j}(re^{i\theta}, \xi), \tag{2.75}$$

where $k \leqslant l$. But such U can satisfy (2.74) for at most $k-1$ arguments unless the $a_j(\xi)$ vanish. It follows that $U=0$. ∎

3. Boundary Values of Analytic Functions

In this section we shall study the basic properties of those analytic functions $F(re^{i\theta})$ that have a distributional boundary value as $r \to 1$.

Let $F(z)$ be analytic for $|z|<1$, let $U(z)$ and $V(z)$ be its real ad imaginary parts. Since $d(F)=d(U)=d(V)$, it follows that if one of these functions is regular harmonic, then so are the other two.

If $u \in \mathscr{D}'(S)$, we shall use the following notation:

(a) $\Lambda(u)$ denotes the harmonic function whose boundary value is u.

(b) $A(u)$ denotes the analytic function whose real part is $\Lambda(u)$ and whose value at 0 is real.

(c) u^* denotes the Hilbert transform of u, that is, the boundary value of the imaginary part of $A(u)$.

Our first result is an extension of the Phragmén–Lindelöf maximum principle.

THEOREM 10. *Let F be a regular harmonic analytic function. Suppose that there exists a closed denumerable set E and a constant M such that*

$$\varlimsup_{z \to \xi} |F(z)| \leqslant M, \qquad \xi \in S \backslash E. \tag{3.1}$$

Then

$$|F(z)| \leqslant M, \qquad |z|<1. \tag{3.2}$$

Proof. Let u and v be as above. We can write $u=u_1+u_2$ where u_1 is bounded by M and where u_2 is a sum of delta functions at points of E. But $A(u_2)$ will then be of the form $\sum_{\xi \in E} p_\xi(1/(z-\xi))$ for certain polynomials p_ξ and such a function is not bounded in $S \backslash E$ unless it reduces to a constant. It follows that F is bounded by M in all S and hence in all B. ∎

The class of regular harmonic analytic functions is closed under sums and products. It is thus a topological algebra, in fact an integral domain. Functions of these class need not be invertible in the algebra, even if they are zero-free in $|z|<1$. An example is provided by the function

$$F(z)=e^{(z+1)/(z-1)}, \tag{3.3}$$

which is bounded by 1 in B, but whose multiplicative inverse

$$G(z) = e^{(z+1)/(1-z)} \tag{3.4}$$

is not regular harmonic. The function G is bounded by 1 on $S \backslash \{1\}$, but not inside B.

Zero-free harmonic functions have some permanence properties, however.

THEOREM 11. *Let F be a zero-free regular harmonic analytic function. Then* $\log F$ *is regular harmonic.*

Proof. Let $W(z) = \operatorname{Re}(\log F(z)) = \log |F(z)|$. There exist constants K and β such that $|F(z)| \leqslant K(1-|z|)^{-\beta}$ and hence

$$W(z) \leqslant \log K(1-|z|)^{-\beta} \leqslant K_1(1-|z|)^{\alpha} \tag{3.5}$$

for every $\alpha > 0$ for suitable constants $K_1 = K_1(\alpha)$. By Theorem 5, W is regular harmonic and hence so is $\log F$. ∎

Recalling the proof of Theorem 5 we see that $C_1(W) = 0$ and so $C_\infty(W) \leqslant d(W) + 1 \leqslant C_1(W) + 1 = 1$. Thus, we immediately obtain

COROLLARY. *Let F be a zero-free regular harmonic analytic function. Then for every $\alpha > 1$ there exists a constant $N = N(\alpha)$, such that*

$$|F(z)| \geqslant e^{N(1-r)^{-\alpha}}. \quad ∎ \tag{3.6}$$

Our class of functions is much larger than the Hardy classes H^p $(p > 0)$. There are zero-free regular harmonic functions that are not in any H^p. We recall that if $F \in H^p$, then the radial pointwise limit $\lim_{r \to 1} F(re^{i\theta})$ exists almost everywhere. We also need the following result of Littlewood [10].

LEMMA 4. *Let (a_n) be a sequence of complex numbers such that*

$$\sum_{n=0}^{\infty} |a_n|^2 = \infty, \tag{3.7}$$

then for almost every choice of signs $\varepsilon_n = \pm 1$, the function

$$F(z) = \sum_{n=0}^{\infty} \varepsilon_n a_n z^n, \tag{3.8}$$

has a radial limit almost nowhere. ∎

Let us now select an increasing sequence of integers (n_k) such that

$$\sum_{k=0}^{\infty} r^{n_k} \leqslant \log\left(\frac{1}{1-r}\right), \qquad \tfrac{1}{2} \leqslant r < 1, \tag{3.9}$$

and let us select the signs $\varepsilon_k = \pm 1$ such that $G(z) = \sum_{k=0}^{\infty} \varepsilon_k z^{n_k}$ has a radial limit almost nowhere. Since

$$|G(z)| \leqslant \log\left(\frac{1}{1-|z|}\right), \tag{3.10}$$

the function $F(z) = e^{G(z)}$ is a zero-free regular harmonic function that belongs to no H^p.

The order of a distribution u is the least integer k such that $u \in (C^k(S))'$. Distributions of order zero are just measures. If the order of u is k, then we can write $u = t^{(k)} + c$ for some measure t and some constant c.

THEOREM 12. *Let U be a regular harmonic function and let k be the order of its boundary value. Then*

$$d(U) \leqslant k \leqslant \max\{d(U) + \tfrac{3}{2}, 0\}. \tag{3.11}$$

Proof. Let u be of order k and let $u = t^{(k)} + c$ where t is a measure. Observe that $d(U) = d(T) + k$ where $T = A(t)$. Since t is a measure there is a constant N such that

$$|\langle t, \phi \rangle| \leqslant N \|\phi\|_\infty. \tag{3.12}$$

In particular,

$$|\langle t, e^{ik\theta} \rangle| \leqslant N, \tag{3.13}$$

that is, the Fourier coefficients of t are bounded. Hence $d(T) \leqslant 0$ and so $d(U) \leqslant k$.

Conversely, let us first suppose that $d(U) < -\frac{1}{2}$. Then the $M_2(r, U)$ are bounded as $r \to 1$, and so u is in $L^2(S)$. In the general case we write $u = t^{(k)} + c$ where t is a measure. Since t' is not a measure we should have $d(T) \geqslant -\frac{3}{2}$. Therefore, $d(U) = d(T) + k \geqslant k - \frac{3}{2}$. ∎

The above theorem determines the order uniquely only for certain values of d. In fact, if $d < -\frac{1}{2}$ then $k = 0$, or if $n - 1 < d < n - \frac{1}{2}$ for some integer n, then the order must be n. But when d belongs to intervals of the form $n - \frac{1}{2} \leqslant d \leqslant n$, the two orders n and $n + 1$ are possible. An example is provided by $F = A(\delta(\theta))$. We have

$$F(z) = \frac{1}{2\pi} \frac{1+z}{1-z} = P(z) + iQ(z), \tag{3.14}$$

where $P(z)$ is the Poisson kernel and

$$Q(re^{i\theta}) = \frac{r\sin\theta}{\pi(1 - 2r\cos\theta + r^2)} \tag{3.15}$$

is its conjugate. The boundary value of P is $\delta(\theta)$, a measure, while that of Q is $(1/2\pi)\cot(\theta/2)$, which is not locally integrable near $\theta = 0$ and hence has order 1. Clearly $d(P) = d(Q) = d(F) = 1$.

By using Lemma 4 we can get a better understanding of this phenomenom. Let $F(z) = \sum a_n z^n$ be a regular harmonic analytic function with $-\frac{1}{2} < d(F) \leqslant 0$. According to the lemma, for almost all choices of sings $\varepsilon_n = \pm 1$, the function $\sum \varepsilon_n a_n z^n$ has a radial limit almost nowhere and hence must be of order 1. By taking derivatives we then obtain

THEOREM 13. *Let $U(re^{i\theta}) = \sum (a_n \cos n\theta - b_n \sin n\theta)\, r^n$ be a regular harmonic function with $k - \frac{1}{2} < d(U) \leqslant k$ for some integer k. Then for almost all choices of signs $\varepsilon_n = \pm 1$ the function $\sum \varepsilon_n (a_n \cos n\theta - b_n \sin n\theta)\, r^n$ has a boundary value of order $k + 1$.* ∎

4. Boundary Value Problems in $\mathscr{D}'(S)$

Let Σ be a smooth closed curve in $\mathbb{C}$. Let τ measure arc length along Σ in the counterclockwise direction, i.e., $z = z(\tau)$, $0 \leqslant \tau \leqslant L$. Then the unit tangent vector is $T(\tau) = z'(\tau)$ and $T'(\tau) = \kappa(\tau)\, N(\tau)$ where $\kappa(\tau)$ is the curvature and where $N(\tau)$ is the outward unit normal vector. Hence [11],

$$\frac{\delta f}{\delta x} = f'(\tau)\, x'(\tau), \qquad \frac{\delta f}{\delta y} = f'(\tau)\, y'(\tau), \tag{4.1}$$

or

$$\frac{\delta f}{\delta x} + i\frac{\delta f}{\delta y} = f'(\tau)(x'(\tau) + iy'(\tau)) = f'(\tau)\, T(\tau). \tag{4.2}$$

Let F be analytic on $\mathbb{C} \backslash \Sigma$ and $f = [F] = F_+ - F_-$ be its jump across Σ. Since

$$F'(z) = \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}, \tag{4.3}$$

the jump formulas of [11] give

$$[F'] = \left[\frac{\partial F}{\partial x}\right] = \left[\frac{dF}{dn}\right] n_x + \frac{\delta f}{\delta x}, \tag{4.4a}$$

and

$$[F'] = \left[-i\frac{\partial F}{\partial y}\right] = -i\left(\left[\frac{dF}{dn}\right]n_y + \frac{\delta f}{\delta y}\right). \tag{4.4b}$$

When we subtract (4.4b) from (4.4a), use (4.2) and the relation $N(\tau) = -iT(\tau)$, we obtain

$$\left[\frac{dF}{dn}\right] = -if'(\tau), \tag{4.5}$$

$$[F'] = \bar{T}(\tau)\, f'(\tau). \tag{4.6}$$

Let $F = U + iV, f = u + iv$; then relation (4.5) yields

$$\left[\frac{dU}{dn}\right] = v'(\tau), \tag{4.7}$$

$$\left[\frac{dV}{dn}\right] = -u'(\tau). \tag{4.8}$$

Next, we apply the jump relation (4.6) to $F'(z)$ to get

$$[F''] = \bar{T}(\tau)\{\bar{T}(\tau)\, f'(\tau)\}',$$

or

$$[F''] = (\bar{T}(\tau))^2\, \{f''(\tau) + \kappa i f'(\tau)\}. \tag{4.9}$$

Similarly, from (4.5) we have

$$\left[\frac{d^2F}{dn^2}\right] = -\{f''(\tau) + \kappa i f'(\tau)\}. \tag{4.10}$$

We now return to $\Sigma = S$, the unit circle. In this case $z(\tau) = N(\tau) = e^{i\tau}$ and $T(\tau) = ie^{i\tau}$ while $\kappa(\tau) = 1$.

We shall denote by $\mathscr{D}_0'(S)$ the set of distributions that satisfy

$$\langle u, 1\rangle = 0. \tag{4.11}$$

When $u \in \mathscr{D}_0'$ we have

$$A(u^*) = -iA(u), \tag{4.12}$$

and so

$$u^{**} = -u. \tag{4.13}$$

Using formula (4.6) we deduce that

$$A(u') = -izA(u)', \tag{4.14}$$

and since $u' \in \mathscr{D}_0'$, (4.12) gives us

$$A(u'^*) = zA(u)'. \tag{4.15}$$

Observe also that by virtue of (4.7), if $U = H(u)$, then

$$dU/dn = u'^*. \tag{4.16}$$

The harmonic product of $u, v \in \mathscr{D}'$ is defined by

$$[u, v] = A^{-1}(A(u)\,A(v)), \tag{4.17}$$

Observe that on $\mathscr{D}_0'$,

$$[u, v^*] = [u^*, v] = [u, v]^*, \tag{4.18}$$

while if $\lambda \in \mathbb{R}$,

$$[\lambda, u] = \lambda u. \tag{4.19}$$

We shall denote by c_n and s_{n+1} the distributions $\cos n\theta$ and $\sin(n+1)\,\theta$, respectively, for $n = 0, 1, 2,\ldots$. Any $u \in \mathscr{D}'$ can be expanded as

$$u = a_0 c_0 + \sum_{k=1}^{\infty} (a_k c_k + b_k s_k). \tag{4.20}$$

We have $A(c_k) = z^k$, $A(s_k) = -iz^k$. Therefore,

$$c_k^* = s_k, \qquad s_k^* = -c_k, \tag{4.21}$$

$$[c_k, c_n] = c_{k+n}, \qquad [c_k, s_n] = s_{k+n}, \tag{4.22a}$$

$$[s_k, c_n] = s_{k+n}, \qquad [s_k, s_n] = -c_{k+n}. \tag{4.22b}$$

The harmonic product makes $\mathscr{D}'$ an integral domain. Let I_n be the ideal generated by c_n, J_n the one generated by s_n. The elements of I_n are of the form

$$a_n c_n + \sum_{k=n+1}^{\infty} (a_k c_k + b_k s_k),$$

while those of J_n are of the form

$$b_n s_n + \sum_{k=n+1}^{\infty} (a_k c_k + b_k s_k).$$

The harmonic multiplication by c_n is a (1–1) map of $\mathscr{D}'$ onto I_n. It then has an inverse which maps I_n onto $\mathscr{D}'$ and which we denote by c_{-n}. Similarly, s_{-n} is the inverse of harmonic multiplication by s_n, defined on J_n. According to (4.22) we have

$$[c_{-n}, c_k] = c_{k-n}, \qquad [c_{-n}, s_{k+1}] = s_{k+1-n}, \qquad k \geqslant n. \tag{4.23a}$$

$$[s_{-n}, s_k] = c_{k-n}, \qquad [s_{-n}, c_{k+1}] = -s_{k+1-n}, \qquad k \geqslant n. \tag{4.23b}$$

The space $\mathscr{D}'$ has another algebraic operation, the ordinary product of a test function ϕ and a distribution t, defined by

$$\langle \phi t, \psi \rangle = \langle t, \phi\psi \rangle. \tag{4.24}$$

The relation between these two products when ϕ is a trigonometric polynomial is as follows. From the elementary relations

$$2 \cos n\theta \cos k\theta = \cos(n+k)\,\theta + \cos(n-k)\,\theta, \tag{4.25a}$$

$$2 \cos k\theta \sin n\theta = \sin(k+n)\,\theta + \sin(n-k)\,\theta, \tag{4.25b}$$

and

$$2 \sin k\theta \sin \theta = \cos(n-k)\,\theta - \cos(n+k)\,\theta, \tag{4.25c}$$

we obtain

$$2c_n u = [c_n + c_{-n}, u], \qquad u \in I_n, \tag{4.26}$$

and

$$2s_n u = [s_n + s_{-n}, u], \qquad u \in J_n, \tag{4.27}$$

More generally, let

$$\phi(\theta) = A_0 + \sum_{j=1}^{n} (A_j c_j + B_j s_j); \tag{4.28}$$

then

$$2\phi u = [\phi + \hat{\phi}, u], \qquad u \in I_n \cap J_n, \tag{4.29}$$

where

$$\hat{\phi} = A_0 + \sum_{j=1}^{n} (A_j c_{-j} + B_j s_{-j}). \tag{4.30}$$

This analysis then gives the following result.

LEMMA 5. *Let p be a trigonometric polynomial of degree n. Let $P=A(p)$, $Q(z)=z^n(P(z)+P(z^{-1}))$ and $q=A^{-1}(Q)$. Then*

$$[2c_n, pu]=[q,u], \qquad u\in I_n\cap J_n. \quad \blacksquare \tag{4.31}$$

We can use the foregoing analysis to solve some boundary value problems involving harmonic functions. We start with the operator

$$Lu=p_1(\theta)\,u'^*+p_2(\theta)\,u'+p_3(\theta)\,u, \tag{4.32}$$

where p_1, p_2, and p_3 are trigonometric polynomials of degree at most n. The equation $Lu=v$ is a Poincaré condition on u (see [12] for the integral equation approach to this problem). We write

$$u=u_1+u_2, \tag{4.33}$$

$$u_1\in I_n\cap J_n, \qquad u_2=a_0c_0+\sum_{k=1}^{n}(a_kc_k+b_ks_k). \tag{4.34}$$

Let q_1, q_2, and q_3 be constructed as in Lemma 5. Then by multiplying the equation

$$Lu=v \tag{4.35}$$

by $2c_n$ we obtain

$$[q_1,u_1'^*]+[q_2,u_1']+[q_3,u_1]=[2c_n, v-Lu_2], \tag{4.36}$$

which after operating with A reduces to the ordinary differential equation

$$(Q_1(z)-iQ_2(z))\,zF'(z)+Q_3(z)\,F(z)=2z^n(G(z)-T(z)), \tag{4.37}$$

where $F=A(u_1)$, $G=A(v)$ and $T=A(Lu_2)$. In some cases an explicit solution is possible.

Let $\mathscr{H}$ be the space of regular harmonic analytic functions and let $\mathscr{H}_0$ be the subset of $\mathscr{H}$ formed by the functions whose value at 0 is real. It is then clear from the definition of A that (4.37) is to be solved in $\mathscr{H}_0$.

We consider a particular example. Let $p(\theta)$ be of the form

$$p(\theta)=\sum_{k=0}^{n}A_kc_k, \tag{4.38}$$

and let us take $p_1=p$, $p_2=0$, and $p_3=-\gamma$ (a constant). We suppose that $p(\theta)$ has different (simple) zeros $\theta_1,\ldots,\theta_n$ in the interval $]0,\pi[$. Equation (4.35) is then the non-normal equation

$$p(\theta)\,u'^*-\gamma u=v. \tag{4.39}$$

We can take a slight modification of (4.34) in this case:

$$u_1 \in I_n + \langle c_0 \rangle, \qquad u_2 = \sum_{k=1}^{n-1} (a_k c_k + b_k c_k) + b_n s_n. \tag{4.40}$$

Suppose first that $v = u_2 = 0$. Then (4.37) reduces to

$$Q(z)\, F'(z) - 2\gamma z^{n-1} F(z) = 0, \tag{4.41}$$

whose solution is $cF_0(z)$, where

$$F_0(z) = \prod_{k=1}^{n} \left[\frac{\bar{\xi}_k z - 1}{\xi_k z - 1} \right]^{\gamma/ip'^*(\theta_k)}, \tag{4.42}$$

where $\xi_k = e^{i\theta_k}$. Since our solution must be in $\mathscr{H}_0$ we require that $c \in \mathbb{R}$. Observe that $F_0(0) = 1$ and $F_0'(0) = \cdots = F_0^{(n-1)}(0) = \operatorname{Im} F_0^{(n)}(0) = 0$.

In the general case we obtain the solution of (4.39) as

$$\begin{aligned} U(z) = {} & \sum_{k=1}^{n-1} \operatorname{Re}(a_k - ib_k)\, z^k - \operatorname{Re}(ib_n z^n) \\ & + \operatorname{Re} \left[\int_0^z \frac{2F_0(z)\, \omega^{n-1}(G(\omega) - T(\omega))\, d\omega}{Q(\omega)\, F_0(\omega)} + a_0 F_0(z) \right], \end{aligned} \tag{4.43}$$

where $G = A(v)$, $T = A(u_2)$, $u_2 = u_2(a_j, b_j)$, and $a_0, a_1, \ldots, a_{n-1}, b_1, \ldots, b_n$ are arbitrary. This U is the unique regular harmonic function that satisfies (4.39) and the conditions

$$a_k = \frac{1}{k!} \frac{\partial^k U}{\partial x^k}(0, 0), \qquad k = 0, \ldots, n-1, \tag{4.44a}$$

$$b_k = \frac{1}{k!} \frac{\partial^k U}{\partial y^k}(0, 0), \qquad k = 1, \ldots, n. \tag{4.44b}$$

Thus, every real value γ is an eigenvalue of multiplicity $2n$ of the operator $L_0 u = pu'^*$. Related non-normal problems are considered in [13].

As a second example we consider the case when $p_1 = \cos\theta - \lambda$, $\lambda > 1$, $p_2 = 0$ and $p_3 = -\gamma$ (a constant). The equation is

$$(c_1 - \lambda)\, u'^* - \gamma u = v. \tag{4.45}$$

We write $u = u_1 + bs_1$, $b = \langle u, s_1 \rangle$, and apply (4.39) to get

$$(z^2 - 2\lambda z + 1)\, F'(z) - 2\gamma F(z) = 2G(z) + biT(z), \tag{4.46}$$

where $F = A(u_1)$, $G = A(v)$ and $T(z) = z^2 - 2(\lambda + \gamma)\, z$. We are looking for solutions in $\mathscr{H}_0$. Let $\tau > 0$ be such that $\lambda = \cosh \tau$, so that $\xi = e^{\tau}$ and $1/\xi = e^{-\tau}$ are the roots of $z^2 - 2\lambda z + 1 = 0$.

We first consider the homogeneous equation associated with (4.46). It has solution in $\mathscr{H}$ if and only if $\lambda = -n \sinh \tau$ where $n \in \mathbb{N}$, those solutions being of the form $cF_0(z)$, where

$$F_0(z) = \left[\frac{1 - z/\xi}{1 - z\xi}\right]^n. \tag{4.47}$$

Again we require $c \in \mathbb{R}$. If $H(z)$ is in $\mathscr{H}$ we put

$$\varphi_n(H) = \frac{-\xi}{n!} \frac{d^n}{dz^n} \left[H(z)(1 - \xi z)^{n-1}\right]\Big|_{z=\xi}, \tag{4.48}$$

and $\rho_n = \varphi_n(T)$. We have

$$\begin{aligned} \rho_n &= \xi(1 - \xi^2)^{-1}, & n &= 0, \\ &= (1 - \xi)^n \left[(n+1)\,\xi^2 + 1 - n\right], & n &\geqslant 1. \end{aligned} \tag{4.49}$$

Let $\beta = (1 + \xi^2)(1 - \xi^2)^{-1}$. Observe that $\rho_n = 0$ only if $n = \beta$. Since (4.46) has solutions in $\mathscr{H}$ for a right-hand side $H(z)$ iff $\varphi_n(H) = 0$, we obtain

Case I. $\gamma = -n \sinh \tau$, $n \in \mathbb{N}$, $n \neq \beta$. Solution of (4.47) exists iff $\operatorname{Re} \varphi_n(G) = 0$, the solution being

$$U(z) = b \operatorname{Im} z + \operatorname{Re}\left[\int_0^z \frac{F_0(z)[2G(\omega) + ibT(\omega)]\, d\omega}{F_0(\omega)(\omega^2 - 2\lambda\omega + 1)} + cF_0(z)\right], \tag{4.50}$$

where $b = -2 \operatorname{Im} \varphi_n(G)\, \rho_n^{-1}$ and $c \in \mathbb{R}$ is arbitrary.

Case II. $\lambda = -n \sinh \tau$, $n \in \mathbb{N}$, $n = \beta$. Solution of (4.45) exists iff $\varphi_n(G) = 0$, the solution being as (4.50) above, but here both $b, c \in \mathbb{R}$ are arbitrary.

Let now $\alpha = -\gamma/\sinh \tau$ be not in $\mathbb{N}$. Then for any right-hand side $H(z)$ in $\mathscr{H}$ there is a unique solution in $\mathscr{H}$ given for $\alpha < n + 1$ by

$$\begin{aligned} F_\alpha(H, z) = (1 - \xi z)^{-\alpha} &\left[\sum_{j=0}^{n} \frac{\rho_j(H, \alpha)(z - \xi)^j}{j - \alpha}\right. \\ &- \frac{(1 - z/\xi)^\alpha}{\xi} \int_\xi^z \left[(-\xi)\, H(\omega)(1 - \xi\omega)^{\alpha - 1}\right. \\ &\left.\left. - \sum_{j=0}^{n} \rho_j(H, \alpha)(\omega - \xi)^j\right]\left(1 - \frac{\omega}{\xi}\right)^{-\alpha+1} d\omega\right], \end{aligned} \tag{4.51}$$

where

$$\rho_j(H, \alpha) = \frac{-\xi}{j!} \frac{d^j}{dz^j} [H(z)(1 - \xi z)^{\alpha - 1}] \bigg|_{z = \xi}. \tag{4.52}$$

Since our solution has to be in $\mathscr{H}_0$, we need to have $F_\alpha(H, 0) \in \mathbb{R}$. We introduce the function $\psi(\alpha) = F_\alpha(T, 0)$, whose zeros (α_k) give the other values $\gamma_k = -\alpha_k \sinh \tau$ for which (4.45) has more than one solution. We remark that $\psi(\alpha)$ is positive for $\alpha < 0$ and that it has simple poles at all the integers (except at β if $\beta \in \mathbb{N}$) with residues $r_n = (-1)^{n+1} \xi^n \rho_n$. Since r_n is positive for $1 \leqslant n < \beta$ and negative for $n = 0$ or $n > \beta$, it follows that $\psi(\alpha)$ has at least one zero in each of the intervals $[n, n+1]$ for all integers $n \geqslant 1$, except possibly for n the integral part of β and $n = \beta - 1$ if $\beta \in \mathbb{N}$. We then have

Case III. $\gamma = -\alpha \sinh \tau$, $\alpha \notin \mathbb{N}$, $\alpha \neq \alpha_k$. Solution of (4.45) exists for any $v \in \mathscr{D}'$, the solution being

$$U(z) = b \operatorname{Im} z + \operatorname{Re}[F_\alpha(2G + ibT, z)],$$
$$\text{where } b = -2 \operatorname{Im}[F_\alpha(G, 0)/\psi(\alpha)]. \tag{4.53}$$

Case IV. $\gamma = -\alpha_k \sinh \tau$. Solution of (4.45) exists iff $\operatorname{Im} F_\alpha(G, 0) = 0$, the solution being (4.53) above where b is arbitrary.

5. Complex Boundary Value Problems

The method of the previous section can be extended to equations involving complex distributions by taking real and imaginary parts. A systematic way of doing this is now discussed.

The harmonic product is extended to the space of complex distributions $\mathscr{D}'(S, \mathbb{C})$ by

$$[u_1 + iu_2, v_1 + iv_2] = [u_1, v_1] - [u_2, v_2] + i\{[u_1, v_2] + [u_2, v_1]\}. \tag{5.1}$$

Let Φ_2 be the set of 2×2 complex matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \qquad a, b \in \mathbb{C} \tag{5.2}$$

Let Φ: $\mathbb{C}^2 \to \Phi_2$ be given by

$$\Phi(a, b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{5.3}$$

Observe that Φ is a linear isomorphism between $\mathbb{C}^2$ and Φ_2 and that

$$\Phi(a, b)\,\Phi(c, d) = \Phi(ac - bd, ad + bc), \tag{5.4}$$

so that Φ_2 is a commutative ring; for our purposes it is convenient to consider Φ_2 as a real algebra.

We shall identify the complex number z with the diagonal matrix $\Phi(z, 0)$. We shall denote by $\hat{\imath}$ the matrix $\Phi(0, 1)$ and by

$$A(z) = \operatorname{Re} z + \hat{\imath} \operatorname{Im} z. \tag{5.5}$$

Since $\hat{\imath}^2 = 1$, both $i + \hat{\imath}$ and $i - \hat{\imath}$ are singular matrices; we shall denote by K_1 and K_2, respectively, the ideals generated by them in the ring Φ_2.

LEMMA 6. (a) *An element $\Phi(a, b)$ is invertible in the ring Φ_2 if and only if $a^2 + b^2 \neq 0$.*

(b) $\Phi_2 = K_1 + K_2$, $K_1 K_2 = 0$.

(c) *Every non-invertible element of Φ_2 is either in K_1 or in K_2. Moreover, the elements of K_1 (resp. K_2) are of the form $A(z)(i + \hat{\imath})$ (resp. $A(z)(i - \hat{\imath})$) for a unique $z \in \mathbb{C}$.*

Proof. If $\det \Phi(a, b) = d \neq 0$, then $[\Phi(a, b)]^{-1} = \Phi(a/d, -b/d)$ is also in Φ_2. It follows that $\Phi(a, b)$ is invertible if and only if its determinant $d = a^2 + b^2$ does not vanish. Statement (b) is clear. Since $\Phi(a, b)(i + \hat{\imath}) = A(a + ib)(i + \hat{\imath})$, any element of K_1 is of the form $A(z)(i + \hat{\imath})$. Finally, let $X \in \Phi_2$, there exist $z_1, z_2 \in \mathbb{C}$ with

$$X = A(z_1)(i + \hat{\imath}) + A(z_2)(i - \hat{\imath}) = \Phi(i(z_1 - z_2), z_1 + z_2). \tag{5.6}$$

Then $\det X = 4 z_1 z_2$ and so X is non-invertible only if $z_1 = 0$ or $z_2 = 0$. ∎

The map $A\colon \mathscr{D}' \to \mathscr{H}_0$ can be extended to $\mathscr{D}'(S, \mathbb{C})$ by

$$\hat{A}(u + iv) = \begin{pmatrix} A(u) & -A(v) \\ A(v) & A(u) \end{pmatrix} = \Phi(A(u), A(v)). \tag{5.7}$$

Comparison of (5.1) and (5.4) gives

$$\hat{A}([t_1, t_2]) = \hat{A}(t_1)\,\hat{A}(t_2). \tag{5.8}$$

We observe that if t is a real distribution then $\hat{A}(t)$ is the diagonal matrix $\Phi(A(t), 0)$ which was identified with the complex function $A(t)$; when z is a complex number then $A(z)$ and $\hat{A}(z)$ coincide. It is therefore natural to eliminate the "hat" when no confusion arises.

If $\mathscr{H}(\Phi_2)$ is the set of regular harmonic analytic functions with values in Φ_2 and $\mathscr{H}_0(\Phi_2)$ the subset of those whose value at 0 is real, then A maps

$\mathscr{D}'(S, \mathbb{C})$ onto $\mathscr{H}_0(\Phi_2)$. Observe that formulas (4.14)–(4.16) remain valid in this case.

EXAMPLE 1. We consider the non-normal equation

$$c_1 u'^* - \gamma u = v, \tag{5.9}$$

where γ is a complex constant and v is a given complex distribution. We write $u = u_1 + bs_1$, $b = \langle u, s_1 \rangle$. Let $F = A(u_1)$ and $G = A(v)$. If we multiply harmonically with $2c_1$ and operate with A we obtain

$$(z^2 + 1) F'(z) - 2A(\gamma) F(z) = 2G(z) + A(b)\, i(z^2 - A(\gamma)\, z). \tag{5.10}$$

The general solution of this equation in $\mathscr{H}_0(\Phi_2)$ is

$$F(a, b, z) = \left[\int_0^z \frac{\{2G(\omega) + A(b)\, i(\omega^2 - 2A(\gamma)\, \omega)\}\, d\omega}{F_0(\omega)(\omega^2 + 1)} + a\right] F_0(z), \tag{5.11}$$

where

$$F_0(z) = \left(\frac{1 - iz}{1 + iz}\right)^{iA(\gamma)}, \tag{5.12}$$

and a is a real matrix in Φ_2. We have used the notation

$$z^A = e^{A \ln z} = \sum_{n=0}^{\infty} \frac{A^n (\ln z)^n}{n!}; \qquad z \in \mathbb{C}, \quad A \in \Phi_2. \tag{5.13}$$

The solution of our original equation (5.9) can then be expressed as

$$U(z) = b \operatorname{Im} z + H(A^{-1}(F(a, b, z))), \tag{5.14}$$

where a is an arbitrary real element of Φ_2 and b is an arbitrary complex number.

EXAMPLE 2. We consider the Poincaré condition

$$au'^* + bu' = cu + v, \tag{5.15}$$

where a, b, c are complex constants and $(a, b) \neq (0, 0)$. Let $F = A(u)$ and $G = A(v)$. Then by operating with A we obtain

$$[A(a) - iA(b)]\, zF'(z) = A(c)\, F(z) + G(z). \tag{5.16}$$

We write $F(z) = \sum_0^\infty \alpha_n z^n$, $G(z) = \sum_0^\infty \beta_n z^n$ to get

$$[A(na - c) - iA(nb)]\, \alpha_n = \beta_n. \tag{5.17}$$

Let $N_+ = (a+ib)\,\mathbb{N}$ and $N_- = (a-ib)\,\mathbb{N}$ and let $A_n = A(na-c) - iA(nb)$. We have the following cases.

Case 1. $c \notin N_+$, $c \notin N_-$. Then A_n is invertible for all n and (5.17) can be solved. Hence

$$F(z) = \sum_0^\infty A_n^{-1} \beta_n z^n. \tag{5.18}$$

Case 2. $c \in N_+$, $c \notin N_-$. Then $c = n(a+ib)$ for some $n \in \mathbb{N}$ and so $A_n \in K_2$. It follows that αz^n, $\alpha \in K_1$ is a solution of the homogeneous equation and so in $\mathscr{D}'(S, \mathbb{C})$ it has the form $w(c_n + is_n)$, $w \in \mathbb{C}$. Solution of the non-homogeneous equation exists if and only if $\beta_n \in K_2$, i.e., if $\langle v, c_n + is_n \rangle = 0$.

Case 3. $c \notin N_+$, $c \in N_-$. Let $n = c(a-ib)^{-1}$. Then $A_n \in K_1$. Solution of the homogeneous equation is $w(c_n - is_n)$, $w \in \mathbb{C}$. Solution of the non-homogeneous equation exist iff $\langle v, c_n - is_n \rangle = 0$.

Case 4. $c \in N_+$, $c \in N_-$, $c \neq 0$. Solution of the homogeneous equation is $w_1 c_n + w_2 s_n$, $w_1, w_2 \in \mathbb{C}$. Solution exists iff $\langle v, s_n \rangle = \langle v, c_n \rangle = 0$.

Case 5. $c = 0$, $a^2 + b^2 \neq 0$. Solution of the homogeneous equation is wc_0. Solution exists iff $\langle v, c_0 \rangle = 0$.

Case 6. $c = 0$, $a = ib$. Solution of homogeneous equation $w_0 c_0 + \sum_1^\infty w_n(c_n - is_n)$. Solution exists iff $v = \sum_1^\infty \lambda_n(c_n + is_n)$.

Case 7. $c = 0$, $a = -ib$. Solution of homogeneous equation $w_0 c_0 + \sum_1^\infty w_n(c_n + is_n)$. Solution exists iff $v = \sum_1^\infty \lambda_n(c_n - is_n)$.

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